MINIMUM PHASE BEHAVIOR OF RANDOM MEDIA

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Abstract

We give two different sufficient conditions for the transfer function of two-mode random media to be minimum phase and illustrate these results with a few examples.

Introduction

Consider the coupled line equations

$$I_0'(z) = -\Gamma_0 I_0(z) + jc(z) I_1(z)$$
$$I_1'(z) = jc(z) I_0(z) - \Gamma_1 I_1(z) \qquad (1)$$

describing a system of two coupled modes traveling in the +z direction. Γ_0 and Γ_1 are the complex propagation constants, with real and imaginary parts

$$\Gamma_0 \equiv \alpha_0 + j\beta_0 \ , \ \Gamma_1 \equiv \alpha_1 + j\beta_1 \ , \qquad (2)$$

and $'$ denotes differentiation with respect to z. $I_0(z)$ and $I_1(z)$ are complex wave amplitudes representing signal and spurious modes respectively, each having time dependence $\exp(j2\pi ft)$. $c(z)$ is a real coupling coefficient having arbitrary functional dependence on distance z; $c(z)$ is taken as a random process with known statistics in some problems.

We assume initial conditions

$$I_0(0) = 1, \ I_1(0) = 0. \qquad (3)$$

Thus, a unit signal is injected in the desired mode at $z = 0$; the output $I_0(z)$ is then the complex signal transfer function for length z of guide.

The following normalization is convenient [1]:

$$I_0(z) \equiv \exp(-\Gamma_0 z) \cdot G_0(z) \qquad (4)$$
$$I_1(z) \equiv \exp(-\Gamma_1 z) \cdot G_1(z) \qquad (4)$$

$$\Delta\Gamma \equiv \Gamma_0 - \Gamma_1 = \Delta\alpha + j\Delta\beta \qquad (5)$$

$$\Delta\alpha = \alpha_0 - \alpha_1$$
$$\Delta\beta = \beta_0 - \beta_1 \qquad (6)$$

Then (1) becomes

$$G_0'(z) = jc(z)\exp(-\Delta\Gamma z) \cdot G_0(z)$$
$$G_1'(z) = jc(z)\exp(+\Delta\Gamma z) \cdot G_0(z) \qquad (7)$$

governing the normalized transfer functions G_0 and G_1. The initial conditions (3) become

$$G_0(0) = 1 \ , \ G_1(0) = 0 \ . \qquad (8)$$

Now let $G_0(\Delta\Gamma)$ be the solution to (7) and (8) for some fixed guide length z and coupling function c(z). Then the normalized signal transfer function for a guide with particular values of attenuation and phase constants is found by substituting (6) for the real imaginary parts of the complex parameter $\Delta\Gamma$ of (5).

The physical applications of these equations have been discussed in several places, in particular in Section I of [1], which gives earlier pertinent references. The following facts, from Sections II and III and Appendix I of [1], are of direct interest here:

1. The signal mode is assumed to have lower heat loss and greater group velocity than the spurious mode:

$$\Delta\alpha = \alpha_0 - \alpha_1 \leq 0 \ . \qquad (9)$$

$$\frac{d}{df}\Delta\beta < 0 \ . \qquad (10)$$

2. Over narrow bands of interest we neglect the frequency dependence of $\Delta\alpha$ and $c(z)$, and assume $\Delta\beta$ varies approximately linearly with frequency f (with negative slope).

Item 2 suggests the substitution

$$-\Delta\beta \rightarrow \lambda \qquad (11)$$

where λ is normalized angular frequency, since

$$\lambda \approx \text{constant} \cdot 2\pi f \qquad (12)$$

over a suitably narrow band. Introduce the complex frequency

$$s \equiv \sigma + j\lambda \ , \qquad (13)$$

as with the LaPlace transform. Then from Sections IV and V and Appendix II of [1],

$$G_0(\Delta\alpha - s) = G_0(\Delta\alpha - \sigma - j\lambda) \qquad (14)$$

gives the behavior of the transfer function G_0 throughout the complex frequency plane, where $G_0(\Delta\Gamma)$ is the solution to (7) and (8) for some fixed guide length.

Therefore let us relate the two complex $\Delta\Gamma$- and s-planes by

$$\Delta\Gamma = \Delta\alpha - s \ , \qquad (15)$$

where $\Delta\alpha$ is the particular value of differential attenuation under consideration. The right-half s-plane,

$$\sigma > 0 \ , \qquad (16a)$$

corresponds to the region

$$\text{Re}\,\Delta\Gamma < \Delta\alpha = -|\Delta\alpha| \qquad (16b)$$

in the $\Delta\Gamma$-plane. The imaginary axis in the s-plane,

$$\sigma = 0 \ , \ s = j\lambda \ , \qquad (17a)$$

corresponds to the vertical line

$$\Delta\Gamma = \Delta\alpha - j\lambda = \Delta\alpha + j\Delta\beta \qquad (17b)$$

in the $\Delta\Gamma$-plane, and G_0 evaluated at points along (17-b) gives the transfer function for sinusoidal inputs, the only values of direct physical interest.

A number of general properties of G_0 were given in Section IV of [1] for arbitrary $c(z)$, valid without perturbation or any other approximations. For example, G_0 is analytic for all finite $\Delta\Gamma$ (and hence for all finite s); i.e., G_0 has no poles or other singularities anywhere in the finite $\Delta\Gamma$- or s-planes. However, [1] contained no information about zeros of G_0. Such information is contained in the following results:

If

$$\int_0^z |c(x)|\,dx < \cosh^{-1} 2 \approx 1.317 \qquad (18)$$

then all zeros of G_0 lie in the region of the s-plane

$$\sigma \equiv \text{Re}\ s < \Delta\alpha \leq 0 \ . \qquad (19)$$

G_0 is consequently minimum phase, since all zeros lie in the left-half s-plane.

If

$$\int_0^s |c(x)|\,e^{\Delta\alpha(s-x)}\,dx < \sqrt{\frac{\sqrt{2}-1}{2}} \approx 0.455$$

$$\text{for} \quad 0 \leq s \leq z \ , \ \Delta\alpha \leq 0 \ , \qquad (20)$$

then G_0 is minimum phase, i.e., all zeros lie in the region of the s-plane

$$\sigma \equiv \text{Re}\ s < 0 \ . \qquad (21)$$

Series Solutions and Minimum Phase

A general series solution for G_0 of (7) and (8), analytic for all finite $\Delta\Gamma$, is given in [1] and [2]. From Appendix II of [1]

$$G_0(\Delta\Gamma) = 1 + \sum_{n=1}^{\infty} (-1)^n G_{0(n)}(\Delta\Gamma) \qquad (22)$$

where the terms are bounded by

$$\left|G_{0(n)}(\Delta\Gamma)\right| \le \frac{\left[\int_0^z |c(x)|dx\right]^{2n}}{(2n)!} \text{ , Re } \Delta\Gamma \le 0 \text{ .} \qquad (23)$$

The precise form for $G_{0(n)}(\Delta\Gamma)$, and bounds for Re $\Delta\Gamma \ge 0$, are given in [1], but are not of interest here. The $G_{0(n)}$ of (22) are in general complex. Then

$$G_0(\Delta\Gamma) \ne 0 \quad \text{if} \quad \left|\sum_{n=1}^{\infty} (-1)^n G_{0(n)}(\Delta\Gamma)\right| < 1. \qquad (24)$$

Using (23)

$$\left|\sum_{n=1}^{\infty} (-1)^n G_{0(n)}(\Delta\Gamma)\right| \le \sum_{n=1}^{\infty} \frac{\left[\int_0^z |c(x)|dx\right]^{2n}}{(2n)!}$$

$$= \cosh\left[\int_0^z |c(x)|dx\right] - 1 \qquad (25)$$

Therefore $G_0(\Delta\Gamma)$ has no zeros in the left-half $\Delta\Gamma$-plane if

$$\cosh\left[\int_0^z |c(x)|dx\right] < 2 \qquad (26)$$

or

$$\int_0^z |c(x)|dx < \cosh^{-1}2 \approx 1.317 \text{ .} \qquad (27)$$

By (16), the condition of (27) and (18) excludes zeros from the right-half s-plane, guaranteeing that G_0 is minimum phase. Moreover, zeros are excluded from a strip extending from Re s = 0 to Re s = $\Delta\alpha < 0$ in the left-half s-plane as stated in (19); consequently the condition of (18) or (27) is stronger than required for G_0 to be minimum phase.

Define the complex signal loss as [2]

$$\Lambda(\Delta\Gamma) = \ln G_0(\Delta\Gamma) \text{ .} \qquad (28)$$

A series solution for Λ was given in [2]. In the region

$$\text{Re } \Delta\Gamma < \Delta\alpha < 0 \qquad (29)$$

this series converges, and hence Λ is analytic, if the condition

$$\int_0^z |c(x)| e^{\Delta\alpha(s-x)} dx < \sqrt{\frac{\sqrt{2} - 1}{2}} \approx 0.455$$

for (30)

$$0 \le s \le z \text{ , } \Delta\alpha \le 0 \text{ ,}$$

given in (20), is satisfied. Since poles of Λ correspond to zeros and poles of G_0, and since G_0 has no finite poles, absence of poles for Λ implies absence of zeros for G_0. From (29) and (16), the condition of (20) or (30) excludes zeros from the right-half s-plane, as stated in (21), and guarantees G_0 to be minimum phase.

The condition of (18) permits larger coupling c(x) than that of (20) for $\Delta\alpha = 0$; for large enough $|\Delta\alpha|$ and most reasonable c(x) the reverse is true. If a given c(x) satisfies both constraints (18) and (20), (19) provides a stronger constraint on the transfer function zeros than (21). In a rough sense, (20)-(21) state that the transfer function is minimum phase if the spurious mode is dissipated faster than it is coupled from the signal mode.

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References

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